HYPERFLUID — A MODEL OF
CLASSICAL MATTER WITH HYPERMOMENTUM

by

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Abstract

A variational theory of a continuous medium is developed the elements of which carry momentum and hypermomentum (*hyperfluid*). It is shown that the structure of the sources in metric-affine gravity is predetermined by the conservation identities and, when using the *Weyssenhoff ansatz*, these explicitly yield the hyperfluid currents.

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1. Introduction

Fluid models (see, e.g., [1-3]) play an important role in gravitational theory, providing a convenient description of classical matter in terms of hydrodynamical notions. Various applications can be mentioned, starting ¿from cosmology and astrophysics and including approximation schemes of the post-Newtonian formalism.

Spin fluids [4-8], or continuous media with internal angular momentum, form the basis for understanding the physics of polarizable matter. This type of a classical (i.e., non-quantum) source occurs most naturally within the framework of Poincaré gauge gravity [9,10].

A further generalization leads to metric-affine gravity based on the gauge theory for the general affine group $GA(4, R)$ [11-17]. This article presents an attempt to construct a variational theory of a *hyperfluid* — a continuous medium the elements of which are characterized by a nontrivial hypermomentum density.

2. Conservation identities and phenomenological approach to a hyperfluid

The gravitational variables of metric-affine gravity are the forms $(g_{\alpha\beta}, \vartheta^\alpha, \Gamma_\alpha{}^\beta)$ with an appropriate transformation behavior under the local $GL(4, R)$ group. The metric $g_{\alpha\beta}$ is a 0-form, the coframe ϑ^α and the connection $\Gamma_\alpha{}^\beta$ are 1-forms. The list of the field strengths includes the *nonmetricity* 1-form $Q_{\alpha\beta} := -Dg_{\alpha\beta}$ besides the 2-forms of torsion T^α and curvature $R_\alpha{}^\beta$ [18, 19].

Let us start the discussion of the generalized fluid by displaying the gravitational field equations. For the most general gravitational Lagrangian 4-form $V = V(g_{\alpha\beta}, \vartheta^\alpha, Q_{\alpha\beta}, T^\alpha, R_\alpha{}^\beta)$ they read:

$$2\frac{\delta V}{\delta g_{\alpha\beta}} = -\sigma^{\alpha\beta}, \tag{2.1}$$

$$\frac{\delta V}{\delta \vartheta^\alpha} = -\Sigma_\alpha, \tag{2.2}$$

$$\frac{\delta V}{\delta \Gamma_\alpha{}^\beta} = -\Delta^\alpha{}_\beta, \tag{2.3}$$

The left-hand sides of the field equations (2.1)-(2.3) are not quite independent. They satisfy the *identities* which result from the Noether theorem for general coordinate (diffeomorphism) and local $GL(4, R)$ invariance of the gravitational action,

$$D\frac{\delta V}{\delta \vartheta^\alpha} \equiv (e_\alpha \rfloor T^\beta) \wedge \frac{\delta V}{\delta \vartheta^\beta} + (e_\alpha \rfloor R_\beta{}^\gamma) \wedge \frac{\delta V}{\delta \Gamma_\beta{}^\gamma} - (e_\alpha \rfloor Q_{\beta\gamma})\frac{\delta V}{\delta g_{\beta\gamma}}, \tag{2.4}$$

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$$D\frac{\delta V}{\delta \Gamma_\alpha{}^\beta} + \vartheta^\alpha \wedge \frac{\delta V}{\delta \vartheta^\beta} - 2g_{\beta\gamma}\frac{\delta V}{\delta g_{\alpha\gamma}} \equiv 0, \tag{2.5}$$

where the vectors e_α constitute the (anholonomic) *frame* $(e_\alpha \rfloor \vartheta^\beta = \delta_\alpha^\beta)$.

The right-hand sides of the gravitational field equations are represented by the *matter currents*: the metric stress-energy $\sigma^{\alpha\beta}$, the canonical energy-momentum Σ_α, and the hypermomentum $\Delta^\alpha{}_\beta$, the latter is asymmetric in α and β.

In a self-consistent *variational* framework the matter currents should arise quite generally from a matter Lagrangian L as variational derivatives

$$\sigma^{\alpha\beta} := 2\frac{\delta L}{\delta g_{\alpha\beta}}, \quad \Sigma_\alpha := \frac{\delta L}{\delta \vartheta^\alpha}, \quad \Delta^\alpha{}_\beta := \frac{\delta L}{\delta \Gamma_\alpha{}^\beta}. \tag{2.6}$$

However, let us suppose that the precise form of the matter Lagrangian is unknown. To what extent can one determine the structure of the matter currents? Such an approach (which could be called *phenomenological*) proved to be useful in Einstein's general relativity theory, as well as in Poincaré gauge gravity. We will demonstrate that, in fact, the conservation identities provide quite a powerful tool for establishing the theory of a fluid with hypermomentum.

In the absence of an explicit matter Lagrangian L, the standard Noether framework is not available for discussing the symmetry properties of matter. However, now the identities (2.4)-(2.5) play a central role, giving the conditions which provide the mathematical self-consistency of the gravitational theory. Indeed, let us substitute (2.1)-(2.3) into these geometrical identities. As a result we obtain two *equations* for the matter currents,

$$D\Sigma_\alpha = (e_\alpha \rfloor T^\beta) \wedge \Sigma_\beta + (e_\alpha \rfloor R_\beta{}^\gamma) \wedge \Delta^\beta{}_\gamma - \frac{1}{2}(e_\alpha \rfloor Q_{\beta\gamma})\,\sigma^{\beta\gamma}, \tag{2.7}$$

and

$$D\Delta^\alpha{}_\beta + \vartheta^\alpha \wedge \Sigma_\beta - g_{\beta\gamma}\,\sigma^{\alpha\gamma} = 0. \tag{2.8}$$

Unlike (2.4)-(2.5) these are not identically satisfied, but instead should be understood as constraints on the *matter variables* (unspecified as yet) out of which the matter currents are constructed.

As it is clear from (2.2), (2.3), the currents $(\Sigma_\alpha, \Delta^\alpha{}_\beta)$ are associated to the *translational* and *rotational-deformational* (local $GL(4, R)$) gravitational degrees of freedom, respectively. Thus they represent the essential physical constituents of the theory, namely the $GA(4, R)$–matter currents. In contrast, the current $\sigma^{\alpha\beta}$ is a secondary object, which is proved by the fact that equation (2.8) determines the current $\sigma^{\alpha\beta}$, provided Σ_α and $\Delta^\alpha{}_\beta$ are given.

Let us present subsequently the phenomenological description of the hyperfluid. We will consider it as a continuous medium the elements of which are characterized by the density of the classical "charge" of the relevant gauge group — i.e., by the

pair $(P_\alpha, J^\alpha_\beta)$ in our case. These quantities represent the 4-momentum and the hypermomentum of a fluid element, respectively. As usually in hydrodynamics, the 4-velocity vector field u^α is defined by the flow of the fluid. In the language of exterior calculus, it is more convenient to start with a *flow 3-form u* [20]. The components of the velocity are then defined by

$$u_\alpha := e_\alpha \rfloor * u, \tag{2.9}$$

where $*$ is the Hodge dual operator. The 4-velocity is assumed to be a timelike vector with unit length, $u^\alpha u_\alpha = 1$. This translates into the condition

$$*u \wedge u = \eta, \tag{2.10}$$

where the volume 4-form is defined by means of the Hodge dual, $\eta := *1$. The current of a $GA(4, R)$-charged fluid, produced by the flow u, is simply a 3-form $u(P_\alpha, J^\alpha_\beta)$. We will assume that this *phenomenological* current describes the right-hand side of (2.2)-(2.3). Thus the hyperfluid matter current 3-forms are given by

$$\Sigma_\alpha = uP_\alpha, \tag{2.11}$$

$$\Delta^\alpha_\beta = uJ^\alpha_\beta. \tag{2.12}$$

The representation (2.11)-(2.12) (which can be called the *generalized Weyssenhoff ansatz*) proved to be viable both in Einstein theory, and in Poincaré gauge gravity. In the former case only the translations effectively form the gauge symmetry group of space-time. Hence the relevant fluid is characterized by the matter currents $\Sigma_\alpha = uP_\alpha$, $\Delta^\alpha_\beta = 0$, which describe an ordinary structureless continuous medium. In Poincaré gravity, the matter currents are $\Sigma_\alpha = uP_\alpha$, $\Delta^\alpha_\beta = uS^\alpha_\beta$, with $S^{\alpha\beta} = -S^{\beta\alpha}$ (spin), and these describe the Weyssenhoff spin fluid [4,8,20]. The antisymmetric part of the hypermomentum charge in (2.12) represents the spin density, $J_{[\alpha\beta]} := S_{\alpha\beta}$.

Remarkably, the conservation identities (2.7)-(2.8) contain much of the information necessary for establishing the hyperfluid theory. At first, let us use (2.8) and find the structure of the canonical energy-momentum and the metrical stress of the hyperfluid. For this purpose we notice that the antisymmetric part of (2.8),

$$g_{\gamma[\alpha}D(uJ^\gamma_{\beta]}) + \vartheta_{[\alpha} \wedge uP_{\beta]} = 0, \tag{2.13}$$

is easily solved with respect to the 4-momentum,

$$P_\alpha = *(u \wedge \pi_\alpha), \qquad \pi_\alpha = \varepsilon\vartheta_\alpha + 2\vartheta^\beta g_{\gamma[\alpha}\dot{J}^\gamma_{\beta]}, \tag{2.14}$$

where hereafter for any covariant object $\Phi^{\alpha\cdots}_{\beta\cdots}$ the dot denotes $\dot{\Phi}^{\alpha\cdots}_{\beta\cdots} := -*D(u\Phi^{\alpha\cdots}_{\beta\cdots})$, and the scalar $\varepsilon := u^\alpha P_\alpha$ has the meaning of the rest energy density.

The substitution of (2.14) back into (2.11) yields the explicit form of the canonical energy-momentum current of the hyperfluid:

$$\Sigma_\alpha = u * \left[u \wedge (\varepsilon\vartheta_\alpha + 2\vartheta^\beta g_{\gamma[\alpha} \dot{J}^\gamma{}_{\beta]}) \right]. \tag{2.15}$$

In turn, the symmetric part of (2.8) yields the metric stress current

$$\sigma_{\alpha\beta} = \eta \left[\varepsilon u_\alpha u_\beta + u^\lambda u_{(\alpha} g_{\beta)\gamma} \dot{J}^\gamma{}_\lambda + h_{\gamma(\alpha} \dot{J}^\gamma{}_{\beta)} \right], \tag{2.16}$$

where $h^\alpha_\beta := \delta^\alpha_\beta - u^\alpha u_\beta$ is the standard projector on the subspace orthogonal to the 4-velocity u^α.

It is worthwhile to note that, when the nonmetricity is zero (i.e., in a Riemann-Cartan space-time), only spin contributes to the canonical energy-momentum current (2.15). This becomes clear after rewriting π_α in (2.14) as $\pi_\alpha = \varepsilon\vartheta_\alpha + 2\vartheta^\beta \dot{S}_{\alpha\beta} + 2Q_{\gamma[\alpha} J^\gamma{}_{\beta]} * (u \wedge \vartheta^\beta)$.

So far no non-gravitational interaction was assumed between the elements of the fluid. Thus the model above describes the case of *incoherent* (or "dust") matter with hypermomentum. The interparticle interactions manifest themselves in the additional *stress* term in the canonical energy-momentum tensor. Within the framework of the phenomenological approach under consideration, this additional term should be postulated separately. The simplest possibility is to adopt the *ideal* fluid postulate, which states that the stress produced by the interparticle interactions is given by the isotropic pressure p. Hence, the final canonical energy-momentum current of the *ideal hyperfluid* reads

$$\Sigma_\alpha = p * u \wedge (e_\alpha \rfloor u) + u * \left[u \wedge (\varepsilon\vartheta_\alpha + 2\vartheta^\beta g_{\gamma[\alpha} \dot{J}^\gamma{}_{\beta]}) \right]. \tag{2.17}$$

The metric stress current (2.16) is modified then, according to (2.8), by including the stress term $-p h_{\alpha\beta} \eta$.

In the next section we present a self-consistent variational theory of the hyperfluid, constructed along the lines of the early spin fluid models.

3. Variational principle for the ideal hyperfluid

We will construct the model of a hyperfluid in generalizing the variational theory of the Weyssenhoff spin fluid [21]. The motion of the medium will be described, as usual, by its 4-velocity u^α and three vectors b^α_A, $A = 1, 2, 3$, attached to each element of the fluid. However, unlike the *rigid* triad of the spin fluid model (which can only rotate), we will assume, in accordance with the $GA(4, R)$ gauge approach, that the material frame b^α_A is *elastic* — in the sense that it can undergo arbitrary deformations

during the motion of the fluid. This transition from a rigid to a deformable material frame is in fact well known in the elasticity theory as the transition from a Cosserat continuum [22] to the elastic medium *with micro-structure* of Mindlin [23]. Like the fluid velocity, we will describe the material frame by the 3-form b_A, such that $b_A^\alpha := *(b_A \wedge \vartheta^\alpha)$, cf. (2.9).

Wishing to preserve the Weyssenhoff model as a limiting case of the theory under consideration, we will assume that the 4-velocity is normalized and orthogonal to the material frame,

$$*u \wedge u = \eta, \quad *u \wedge b_A = 0. \tag{3.1}$$

Technically, it will be convenient also to introduce explicitly the dual material triad — the material co-frame b_α^A, $A = 1, 2, 3$ which satisfies $b_\alpha^A b_B^\alpha = \delta_B^A$. In the exterior form language this will be described by the 1-form b^A (so that $b_\alpha^A := e_\alpha \rfloor b^A$) which is dual to the 3-form b_A,

$$b^B \wedge b_A = \delta_A^B \eta. \tag{3.2}$$

We will treat the pair (b_A, b^B) as independent dynamical variables and (3.2) as the constraint, which simplifies greatly the non-trivial problem of raising and lowering indices in the metric-affine approach.

The internal structure of the hyperfluid is characterised by the following scalar variables: the particle density ρ, the specific entropy s, the Lin variable X [24,25] (used to identify particles), and the specific hypermomentum density μ^A_B which is the direct generalisation of the spin density variable of the old Weyssenhoff model. As usually, we assume that the number of particles is not changed and that the entropy and the identity of particles is conserved during the motion of the fluid. These conditions are manifested in the form of constraints

$$d(\rho u) = 0, \tag{3.3}$$

$$u \wedge dX = 0, \tag{3.4}$$

$$u \wedge ds = 0, \tag{3.5}$$

which will be introduced into the variational principle by means of the method of the Lagrange multipliers.

Now we are in a position to write the Lagrangian 4-form of the hyperfluid:

$$L = \varepsilon(\rho, s, \mu^A_B)\eta - \frac{1}{2}\rho\mu^A_B b_\alpha^B u \wedge Db_A^\alpha -$$

$$-\rho u \wedge d\lambda_1 + \lambda_2 u \wedge dX + \lambda_3 u \wedge ds + \lambda_0(*u \wedge u - \eta)+$$

$$+\lambda^A(*u \wedge b_A) + \lambda_B^A(b^B \wedge b_A - \eta\delta_A^B) + \tilde{\lambda}_A(b^A \wedge u). \tag{3.6}$$

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The first line is most essential physically, representing the internal energy density ε which is assumed to be the function of the particle density, entropy and the specific hypermomentum density, and the kinetic energy (the second term in (3.6)), which represents in fact the sum of well known rotational and elastic deformation energy terms. The remaining terms in (3.6) describe constraints (the last term is necessary to ensure the orthogonality of 4-velocity and the co-frame, since the latter is treated as independent variable). The Lagrange multipliers are 0-forms.

Let us derive the Euler-Lagrange equations. To summarise, the independent variables here are: the metric-affine gravitational field $g_{\alpha\beta}$, ϑ^α, $\Gamma_\beta{}^\alpha$, the material variables b_A, b^B, $\rho, \mu^A{}_B$, s, X, and the Lagrange multipliers λ.

Varying the action with respect to the latter, one obtains the constraints (3.1), (3.2), supplemented by $b^A \wedge u = 0$, and equations (3.3)-(3.5). Variations of s and X yield the equations for the pair of Lagrange multipliers,

$$\eta\Big(\frac{\partial \varepsilon}{\partial s}\Big) + d(\lambda_3 u) = 0, \quad d(\lambda_2 u) = 0, \tag{3.7}$$

while the variations of ρ and $\mu^A{}_B$ yield respectively

$$\eta\Big(\frac{\partial \varepsilon}{\partial \rho}\Big) - \frac{1}{2}\mu^A{}_B b^B_\alpha u \wedge Db^\alpha_A - u \wedge d\lambda_1 = 0, \tag{3.8}$$

$$\eta\Big(\frac{\partial \varepsilon}{\partial \mu^A{}_B}\Big) = \frac{1}{2}\rho b^B_\alpha u \wedge Db^\alpha_A. \tag{3.9}$$

These in fact provide the explicit form of thermodynamical variables: the pressure and the rotation+deformation tensor, conjugated to hypermomentum.

To finish with the material variables, let us write the equations which result from the variation of the action (3.6) with respect to u, b^B, b_A. These read respectively:

$$-\frac{1}{2}\rho\mu^A{}_B b^B_\alpha Db^\alpha_A - \rho d\lambda_1 + \lambda_2 dX + \lambda_3 ds - 2\lambda_0 * u - \lambda^A * b_A - \tilde{\lambda}_A b^A = 0, \tag{3.10}$$

$$\frac{1}{2}\rho\mu^A{}_B * (u \wedge Db^\alpha_A)\eta_\alpha + \lambda^A_B b_A + \tilde{\lambda}_B u = 0, \tag{3.11}$$

$$\frac{1}{2} * D(\rho\mu^A{}_B b^B_\alpha u)\vartheta^\alpha + \lambda^A * u + \lambda^A_B b^B = 0. \tag{3.12}$$

Hereafter, as usually the dual 3-form of the space-time coframe is denoted $\eta^\alpha := *\vartheta^\alpha$.

It is worth to note, that the variations of the matter triad *components* look rather non-trivial:

$$\eta\delta b^\alpha_A = \vartheta^\alpha \wedge \delta b_A + \delta\vartheta^\alpha \wedge b_A - b^\alpha_A\Big(\frac{1}{2}g^{\rho\sigma}\delta g_{\rho\sigma}\eta + \delta\vartheta^\beta \wedge \eta_\beta\Big), \tag{3.13}$$

$$\eta \delta b_\alpha^B = \delta b^B \wedge \eta_\alpha - \delta \vartheta^\beta \wedge \eta_\alpha b_\beta^B. \qquad (3.14)$$

Multiplying (3.10) by u, and using (3.1)-(3.5), (3.7), we get the Lagrange multiplier

$$2\lambda_0 = -\frac{1}{2}\rho\mu^A_{\ B}b_\alpha^B * (u \wedge Db_A^\alpha) - \rho * (u \wedge d\lambda_1) = \rho\Big(\frac{\partial\varepsilon}{\partial\rho}\Big). \qquad (3.15)$$

Analogously, the remaining Lagrange multipliers are obtained from the exterior products of (3.11) with $*u$, (3.11) with b^C, and (3.12) with u, respectively,

$$\tilde{\lambda}_A = -\frac{1}{2}\rho\mu^B_{\ A}u_\alpha * (u \wedge Db_B^\alpha), \qquad (3.16)$$

$$\lambda_B^A = \frac{1}{2}\rho\mu^C_{\ B} * (u \wedge b_C^\alpha Db_\alpha^A), \qquad (3.17)$$

$$\lambda^A = -\frac{1}{2} * D(\rho\mu^A_{\ B}b_\alpha^B u)u^\alpha. \qquad (3.18)$$

Finally, multiplying (3.12) by b_C, and using (3.17), we get the *equations of motion* of the specific hypermomentum density:

$$u \wedge (d\mu^A_{\ B} + \mu^A_{\ C}b_B^\alpha Db_\alpha^C - \mu^C_{\ B}b_C^\alpha Db_\alpha^A) = 0. \qquad (3.19)$$

This generalises the well known equation of motion of spin in the Weysenhoff model.

Let us find the *matter currents*. In the previous section we discussed these phenomenologically, but now the rigorous derivation from the variational principle is straightforward. One obtains:

$$\sigma^{\alpha\beta} := 2\frac{\delta L}{\delta g_{\alpha\beta}} = \eta[\varepsilon g^{\alpha\beta} + 2\lambda_0(u^\alpha u^\beta - g^{\alpha\beta}) + 2\lambda^A b_A^{(\alpha}u^{\beta)} - g^{\alpha\beta}\lambda_A^A - \frac{1}{2}g^{\alpha\beta} * D(\rho\mu^A_{\ B}b_\gamma^B u)b_A^\gamma],$$
$$(3.20)$$

$$\Delta^\alpha_{\ \beta} := \frac{\delta L}{\delta\Gamma_\alpha^{\ \beta}} = \frac{1}{2}\rho\mu^A_{\ B}b_\beta^B b_A^\alpha u, \qquad (3.21)$$

$$\Sigma_\alpha := \frac{\delta L}{\delta\vartheta^\alpha} = \varepsilon\eta_\alpha - 2\lambda_0(\eta_\alpha - u_\alpha u) +$$

$$+u_\alpha\lambda^A b_A + g_{\alpha\beta}b_A^\beta\lambda^A u - \lambda_A^A\eta_\alpha - \frac{1}{2}\eta_\alpha b_A^\beta * D(\rho\mu^A_{\ B}b_\beta^B u) + \frac{1}{2}\eta_\beta * D(\rho\mu^A_{\ B}b_\alpha^B b_A^\beta u). \quad (3.22)$$

These expressions are simplified greatly if we denote

$$J^\alpha_{\ \beta} = \frac{1}{2}\rho\mu^A_{\ B}b_A^\alpha b_\beta^B, \qquad (3.23)$$

and introduce the *pressure* in a standard way,

$$p := \rho\Big(\frac{\partial\varepsilon}{\partial\rho}\Big) - \varepsilon. \qquad (3.24)$$

We then find

$$\lambda_0 = \frac{1}{2}(\varepsilon + p), \tag{3.25}$$

and, finally, the matter currents of the hyperfluid read

$$\sigma^{\alpha\beta} = \eta(\varepsilon u^\alpha u^\beta - p h^{\alpha\beta}) + 2 u^\gamma u^{(\alpha} D\Delta^{\beta)}_{\ \gamma}, \tag{3.26}$$

$$\Delta^\alpha_{\ \beta} = u J^\alpha_{\ \beta}, \tag{3.27}$$

$$\Sigma_\alpha = \varepsilon u u_\alpha - p(\eta_\alpha - u u_\alpha) + 2 u u^\beta g_{\gamma[\alpha} \dot{J}^\gamma_{\ \beta]}. \tag{3.28}$$

In the derivation of these the equations of the hypermomentum (3.19) were used.

Hypermomentum dynamics can be more conveniently described with respect to an anholonomic space-time frame, using the definition (3.23) and the fact that the material frame spans the space, orthogonal to the 4-velocity, which is expressed by the identity

$$b^A_\alpha b^\beta_A = h^\beta_\alpha = \delta^\beta_\alpha - u^\beta u_\alpha, \tag{3.29}$$

straightforwardly derivable from (3.1), (3.2). Multiplying (3.19) by $\frac{1}{2}\rho b^\alpha_A b^B_\beta$, one finds

$$D\Delta^\alpha_{\ \beta} = u^\alpha u_\lambda D\Delta^\lambda_{\ \beta} + u_\beta u^\lambda D\Delta^\alpha_{\ \lambda}. \tag{3.30}$$

Comparison of (3.26)-(3.28) with the expressions of matter currents (2.11), (2.12), (2.17), (2.16), shows complete agreement of the rigorous variational theory with the phenomenological approach. There is, though, one refinement — the hypermomentum density (3.23) is subject to the constraint

$$J^\alpha_{\ \beta} u^\beta = J^\alpha_{\ \beta} u_\alpha = 0, \tag{3.31}$$

which is the analogue of the well known Frenkel condition. Physically this is motivated by the properties of spin as a part of the total hypermomentum.

A new point is that the variational theory yields the equations of motion of hypermomentum (3.30), which the phenomenological approach could not provide in view of the absence of a definition of the phenomenological matter currents in terms of the true dynamical variables of the hyperfluid.

4. Conclusion

The hyperfluid represents a classical model of matter with hypermomentum which is the source of the metric-affine gravity. It is interesting to obtain the exact non-vacuum solutions for the gravitational field equations in astrophysical and cosmological (very early stages) setting, this work is now in progress. Further development of the hyperfluid model should give an answer to an important question: is it possible to recover this medium in a semiclassical treatment of the quantum matter with hypermomentum — such as, for example, the manifields [26-28]?

Acknowledgements

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